A MARKOVIAN REPLACEMENT MODEL WITH A GENERALIZATION TO INCLUDE STOCKING

BY

SHELDON M. ROSS

TECHNICAL REPORT NO. 97

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NONTECHNICAL SUMMARY

The following model is considered: A machine is inspected at the beginning of discrete time periods and its operating state (condition) is determined. After each inspection a decision upon whether or not to replace the machine at the end of the period must be made. If the machine is replaced a new machine must be purchased. It is assumed that it takes one period to purchase and install a new machine — during this period the process is said to be in state 0. If the machine is in operating state (condition) i and a decision not to replace is made then there are known transition probabilities P_{ij} such that the operating state of the machine at time $t+1$ will be j with probability P_{ij}. Thus, for instance, P_{i0} is the probability that a machine in operating state i will fail. If the machine does fail during a period then it is removed at the end of the period and a new machine must then be purchased. As before this corresponds to being in state 0.

It is supposed that each time a machine is classified as being in operating state i an operating cost $C(i)$ is incurred. Also whenever the process is in state 0 a cost $C(0)$ is incurred. This may include both the costs due to buying, delivering, and installing a new machine and also a cost due to the fact that no machine is in use for one period.

A policy is any rule for deciding when to replace the machine and when to leave it alone. It is shown that if (1) $C(i)$ - the operating cost associated with state i - is increasing in i (for $i > 0$) and if (2) the transition into any higher block of states $\{K,K+1,\ldots\}$ is more likely for a higher numbered state then for a lower numbered state,

then the policy which leads to the smallest (long-run) average cost has a very simple form. Its form is that for some integer j it replaces when in operating state i if and only if $i \geq j$.

Methods for finding the optimal policy (or equivalently the critical value j) are discussed and a numerical example is given.

Often in practice there is an additional cost incurred whenever a non-planned replacement — i.e., a replacement caused by a failure — occurs. It is shown that even in this case the optimal policy has the same form as before.

In the last section of this paper the case where more than one machine may be purchased at a given time is considered. Thus when a machine fails many machines may be purchased. One could then be put in use and the others held as reserves. This might be desirable for (1) it might be less expensive to buy the units in quantity, (2) it might cut down on delivery and installation costs, and (3) it would cut down the number of periods during which no machine is in use. It is then shown that a multi-dimensional generalization of the optimal policy for the original (no reserves) problem is optimal for this problem.

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1. Model and Summary of Results.

This paper is concerned with the following countable state Markovian Replacement Model: A unit (piece of equipment, system, etc.) is observed at the beginning of discrete time periods $t = 0,1,2,\ldots$ and classified as being in one of a countable number of states labeled by the non-negative integers. After observing the state of the unit the observer must choose one of two possible actions: Action 1 is to leave the unit in service; Action 0 is to remove the unit from service at the end of the period. It is assumed that a unit in state i will fail during the period before the next observation with probability P_{i0}. If a unit fails it is removed at the end of the period. Whenever a unit is removed from service (either by action 0 or by action 1 and a subsequent failure) a new unit must be purchased. This corresponds to being in state 0. Thus only one action is possible when in state 0 and that is to purchase a new unit. It is assumed that it takes one time period to purchase and install a new unit.

If action 1 is chosen at time t then there are known transition probabilities P_{ij} $i=1,2,\ldots$ $j=0,1,\ldots$ such that

$P\{X_{t+1} = j \mid X_t = i, \Delta_t = 1\} = P_{ij}$ where

X_t = state of the unit in use at time t, and

Δ_t = action chosen at time t.

Of course, if action 0 is chosen at time t then $X_{t+1} = 0$ - i.e.,

$$P(X_{t+1} = 0 | X_t = i, \Delta_t = 0) = 1 \quad \text{for } i=1,2,\ldots$$

Also, when a new unit is purchased there is a kn wn probability distribution $\{P_i\}_{i=0}^{\infty}$ over its initial state - i.e., $P(X_{t+1}=i|X_t=0)=P_i$. Thus, for instance, P_0 may be interpreted as the probability that a new machine will be inoperative.

We shall call X_t the state of the process at time t. Each time the process is in state i an expected cost $C(i)$ is incurred. Thus, for $i > 0$, $C(i)$ may be interpreted as the expected operating cost incurred during one period by a unit which is in state i at the beginning of the period. $C(0)$ includes the cost of buying and installing a new unit; it may also include a cost due to the fact that no unit is in use for that period.

A policy is any rule for choosing actions. In sections 3 and 4 of this paper under suitable conditions on the costs and transition probabilities (given in section 2), the structure of an optimal policy with respect to (1) the discounted-cost and (2) the average-cost criterion is determined. Theorems 3.2 and 4.2 of these sections generalize results given in [2]. This generalization is in two directions. Firstly, we allow for a countable number of states (versus a finite number in [2]) and secondly we allow for a somewhat more general class of transition probabilities. The other theorems in sections 3 and 4 are new and further characterize the structure of the optimal policy. These theorems show that the decision upon whether or not to replace at time t may sometimes be determined solely by the conditional expected cost incurred at time $t + 1$ given that you don't replace and a failure doesn't occur.

For example, it is shown that, under pure deterioration, the optimal average-cost policy replaces only when this conditional expected cost is larger than the optimal average cost per unit time.

In section 5 methods for finding the optimal policy by exploiting its known structure are suggested. A numerical example is given.

In section 6 it is shown how the previous results may be extended to the case where there is a penalty cost incurred whenever a non-planned replacement (i.e., a replacement caused by a failure) occurs.

In section 7 the case where more than one unit may be purchased at a given time is considered and it is shown that optimal policies exist and are analogous in structure to those of sections 3 and 4. This is a joint stocking and replacement model and the results generalize those given in [3].

2. Conditions and Preliminary Lemma

We impose the following conditions on the costs and transition probabilities:

Condition 1: $\{C(i)\}_{i=1}^{\infty}$ is a non-decreasing bounded sequence

Condition 2: $\{P_{i0}\}_{i=1}^{\infty}$ is a non-decreasing sequence

Condition 3: For each $k = 1,2,\ldots$ the function $r_k(i) = 1/1-P_{i0} \sum_{j=k}^{\infty} P_{ij}$ is a non-decreasing function of i for $i = 1,2,\ldots$. (where 0/0 is taken to be ∞).

Thus Conditions 1 and 2 say that the operating cost and failure probability are both non-decreasing functions of the state. Condition 3 says that the conditional probability of a transition into any block of states $\{k,k+1,\ldots\}$, given that action 1 is chosen and a failure does not occur, is a non-decreasing function of the present state i (for $i > 0$).

For notational ease we shall assume throughout that $P_{i0} < 1$ for all i. It is quite easy to show that all of the results still hold even if this is not the case.

The following lemma will be needed. Its proof may be found in [2]

Lemma 2.1: Condition 3 implies that for every non-decreasing bounded sequence $\{h(j)\}_{j=1}^{\infty}$ the function $k(i) = 1/1-P_{i0} \sum_{j\neq 0} P_{ij}h(j)$ is also non-decreasing for $i = 1,2,\ldots$.

3. Discounted-Cost Solution

We are interested in finding a policy R_β such that

$\psi(i,\beta,R_\beta) = \min_R \psi(i,\beta,R)$ for all $i = 0,1,\ldots,$ where

$$\psi(i,\beta,R) = \sum_{t=0}^{\infty} \beta^t E_R[C(X_t) | X_0 = i]$$

and where $0 < \beta < 1$. Such a policy is said to be an optimal β-discount policy or for short, a β-optimal policy.

Let R_β be a β-optimal policy and let $V_\beta(i) = \psi(i,\beta,R_\beta)$ $i = 0,1\ldots$.

<u>Lemma 3.1</u>: Under Conditions 1, 2, 3 $\{V_\beta(i) - C(i)\}_{i=1}^{\infty}$ is a non-decreasing sequence.

Proof: Let $V_\beta(i,1) = C(i)$ for $i = 0,1,\ldots$ and define recursively

(1) $V_\beta(i,n) = \min\{C(i) + \beta \sum_{j=0}^{\infty} P_{ij} V_\beta(j,n-1); C(i) + \beta V_\beta(0,n-1)\}$ for $i > 0$

and

$$V_\beta(0,n) = C(0) + \beta \sum_{j=0}^{\infty} P_j V_\beta(j,n-1)$$

We first show by induction that $\{V_\beta(i,n) - C(i)\}_{i=1}^{\infty}$ is a non-decreasing sequence for each n. For $n = 1$ it follows trivially. Assume then that $\{V_\beta(i,n-1) - C(i)\}_{i=1}^{\infty}$ is non-decreasing and let $0 < i < k$. There are two cases:

<u>Case 1</u>: $V_\beta(i,n) = C(i) + \beta V_\beta(0,n-1)$, which implies by (1) that

(2) $\sum_{j=0}^{\infty} P_{ij} V_\beta(j,n-1) \geq V_\beta(0,n-1)$, or equivalently that

(3) $1/1\text{-}P_{i0} \sum_{j\neq 0} P_{ij} V_\beta(j,n-1) \geq V_\beta(0,n-1)$.

Now $\{V_\beta(i,n-1) - C(i)\}_{i=1}^{\infty}$ nondecreasing implies that $\{V_\beta(i,n-1)\}_{i=1}^{\infty}$ is nondecreasing, and so by Lemma 2.1 and (3) $\kappa > i > 0$ implies

(4) $1/1-P_{k0} \sum_{j\neq 0} P_{kj} \, V_\beta(j,n-1) \geq V_\beta(0,n-1)$, which implies by (1) that

(5) $V_\beta(k,n) - C(k) = \beta \, V_\beta(0,n-1)$

$$= V_\beta(i,n) - C(i) \; .$$

<u>Case 2</u>: $V_\beta(i,n) = C(i) + \beta \sum_{j=0}^{\infty} P_{ij} \, V_\beta(j,n-1)$

and so by (1) we have that

(6) $1/1-P_{i0} \sum_{j\neq 0} P_{ij} \, V_\beta(j,n-1) \leq V_\beta(0,n-1)$, now

(7) $$\sum_{j=0}^{\infty} P_{ij}V_\beta(j,n-1) = P_{i0}V_\beta(0,n-1) + (1-P_{k0})1/1-P_{i0}\sum_{j\neq 0} P_{ij}V_\beta(j,n-1)$$

$$+ (P_{k0}-P_{i0})1/1-P_{i0}\sum_{j\neq 0} P_{ij}V_\beta(j,n-1)$$

and so by Lemma 2.1 and (6) we have that

(8) $$\sum_{j=0}^{\infty} P_{ij}V_\beta(j,n-1) \leq P_{i0}V_\beta(0,n-1) + (1-P_{k0})1/1-P_{k0}\sum_{j\neq 0} P_{kj}V_\beta(j,n-1)$$

$$+ (P_{k0}-P_{i0})V_\beta(0,n-1)$$

$$= \sum_{j=0}^{\infty} P_{kj}V_\beta(j,n-1) \; , \quad \text{and so}$$

(9) $V_\beta(i,n) - C(i) < \beta \sum_{j=0}^{\infty} P_{kj}V_\beta(j,n-1)$.

but from (1) we also have that

(10) $V_\beta(i,n) - C(i) \leq \beta \, V_\beta(0,n-1)$

and so from (9), (10) and (1) we get that

(11) $V_\beta(i,n) - C(i) \leq V_\beta(k,n) - C(k)$

and so $k > i > 0$ implies that $V_\beta(k,n) - C(k) \geq V_\beta(i,n) - C(i)$.

$\therefore$ by induction $\{V_\beta(i,n) - C(i)\}_{i=1}^{\infty}$ is non-decreasing for all n.

Now $V_\beta(i,n)$ is the minimum expected discounted costs incurred over n-stages given that you start in state i. Since $\beta < 1$ and costs are bounded, it is easy to see that $V_\beta(i,n) \to V_\beta(i)$ as $n \to \infty$ for each i.

$\therefore$ $\{V_\beta(i) - C(i)\}_{i=1}^{\infty}$ is non-decreasing.

Q.E.D.

Remark: Since $\{C(i)\}_{i=1}^{\infty}$ is non-decreasing it follows that $\{V_\beta(i)\}_{i=1}^{\infty}$ is non-decreasing.

Definition: A policy R that replaces (takes action 0) at time t iff $X_t \geq j$ for some $j = 1,2,\ldots \infty$ is called a control-limit policy. The control-limit policy with $j = \infty$ is the policy which never replaces.

<u>Theorem 3.2</u>: Under Conditions 1, 2, 3 there is a control-limit policy which is β-optimal.

Proof: It is well known (see [1]) that $V_\beta(i)$ $i = 0,1,2, \ldots$ satisfies

$$V_\beta(i) = \min\{C(i) + \beta \sum_{j=0}^{\infty} P_{ij} V_\beta(j);\ C(i) + \beta V_\beta(0)\} \quad \text{for } i = 1,2,..$$

and any policy which chooses action 1 in state i when the first term is the minimum and action 0 when the second term is minimum is β-optimal. Let $i_\beta = \infty$ if $\sum_{j=0}^{\infty} P_{ij} V_\beta(j) \leq V_\beta(0)$ for all $i > 0$, otherwise

$$\text{let } i_\beta = \min\{i: \sum_{j=0}^{\infty} P_{ij} V_\beta(j) > V_\beta(0)\}$$

$$= \min\{i: 1/1-P_{i0} \sum_{j\neq 0} P_{ij} V_\beta(j) > V_\beta(0)\}$$

$\therefore$ by lemma 2.1 $i \geq i_\beta \Rightarrow 1/1-P_{i0} \sum_{j\neq 0} P_{ij} V_\beta(j) > V_\beta(0)$ or equivalently

that $\sum_{j=0}^{\infty} P_{ij} V_\beta(j) > V_\beta(0)$

$\therefore$ the policy which replaces whenever the process is in state $i \geq i_\beta$ and doesn't whenever in state $i < i_\beta$ is β-optimal.

Q.E.D.

<u>Corollary 3.3</u>: The control-limit policy which replaces in state i if and only if $\sum_{j=0}^{\infty} P_{ij} V_\beta(j) > V_\beta(0)$ is β-optimal.

Proof: Follows from theorem 3.1.

The next theorem further characterizes the structure of the optimal policy.

<u>Theorem 3.4</u>: Under Conditions 1, 2, 3 if i is such that

$$1/1-P_{i0} \sum_{j\neq 0} P_{ij} C(j) \leq (1-\beta)\, V_\beta(0) \qquad (i > 0)$$

then there is a β-optimal control-limit policy which does not replace when the process is in state i. Also, if the above inequality is strict then any policy which replaces at state i is not β-optimal.

Proof: $V_\beta(i) \leq C(i) + \beta\, V_\beta(0)$ for $i = 1, 2, \ldots$

$$\therefore\ 1/1-P_{i0} \sum_{j\neq 0} P_{ij} V_\beta(j) \leq 1/1-P_{i0} \sum_{j\neq 0} P_{ij} C(j) + \beta\, V_\beta(0)$$

Now suppose that $1/1-P_{i0} \sum_{j\neq 0} P_{ij} C(j) \leq (1-\beta)\, V_\beta(0)$

$$\therefore\ 1/1-P_{i0} \sum_{j\neq 0} P_{ij} V_\beta(j) \leq V_\beta(0)$$

$$\therefore \sum_{j=0}^{\infty} P_{ij} V_\beta(j) \leq V_\beta(0)$$

and so the first part of the theorem follows from Corollary 3.3.

To prove the second part suppose that

$$1/1-P_{i0} \sum_{j\neq 0} P_{ij} C(j) < (1-\beta) V_\beta(0)$$

Then, as above, this implies that

$$\sum_{j=0}^{\infty} P_{ij} V_\beta(j) < V_\beta(0)$$

Now let R be any policy which replace at state i

$$\begin{aligned} \therefore \quad \psi(i,\beta,R) &= C(i) + \beta\psi(0,\beta,R) \\ &\geq C(i) + \beta V_\beta(0) \\ &> C(i) + \beta \sum_{j=0}^{\infty} P_{ij} V_\beta(j) \geq V_\beta(i) \end{aligned}$$

and so R is not β-optimal.

Q. E.D.

Often we deal with a process in whcih the unit in use can only deteriorate in time. This is represented mathematically by the following:

Definition: If $P_{ij} = 0$ for all $0 < j < i$ then we call the process a pure deterioration process.

Theorem 3.4: In a pure deterioration process, under Conditions 1, 2, 3, the control-limit policy which replaces at those states i for which $1/1-P_{i0} \sum_{j\neq 0} P_{ij} C(j) > (1-\beta) V_\beta(0)$ is β-optimal.

Proof: Suppose that $1/1-P_{i0} \sum_{j \neq 0} P_{ij} C(j) > (1-\beta) V_\beta(0)$, suppose further that there is an optimal policy which does not replace at state i

$$\therefore \quad V_\beta(i) = C(i) + \beta \sum_{j=0}^{\infty} P_{ij} V_\beta(j)$$

$$= C(i) + \beta P_{i0} V_\beta(0) + \beta \sum_{j \neq 0} P_{ij} C(j) + \beta \sum_{j \neq 0} P_{ij}(V_\beta(j)-C(j))$$

$$> C(i) + \beta P_{i0} V_\beta(0) + \beta(1-P_{i0})(1-\beta)V_\beta(0) + \beta \sum_{j=i}^{\infty} P_{ij}(V_\beta(j)-C(j))$$

$$\geq C(i) + \beta P_{i0} V_\beta(0) + \beta(1-P_{i0})(1-\beta)V_\beta(0) + \beta(1-P_{i0})(V_\beta(i)-C(i))$$

where the last inequality follows from Lemma 3.1.

$$\therefore \quad (1 - \beta + \beta P_{i0})V_\beta(i) > (1 - \beta + \beta P_{i0})(C(i) + \beta V_\beta(0))$$

or $V_\beta(i) > C(i) + \beta V_\beta(0)$

which is a contradiction, and so every optimal policy replaces at state i and the result follows from Theorem 3.4.

Q.E.D.

Thus, in a pure deterioration process, the β-optimal policy replaces at time t whenever the conditional expected cost at time $t + 1$, given that you don't replace and a failure doesn't occur, is greater than $(1-\beta)V_\beta(0)$. The significance and intuitive content of Theorems 3.4 and 3.5 become clearer when we consider them in connection with the average-cost criterion.

4. Average-Cost Solution

For the average-cost criterion we are interested in a policy R^* such that $\varphi(i,R^*) = \min_R \varphi(i,R)$ for all $i = 0,1,2,\dots$

where $$\varphi(i,R) = \lim_{n \to \infty} \sum_{t=0}^{n} E_R[C(X_t)|X_0 = i]/n \ .$$

Such a policy R^*, if it exists, is said to be optimal.

We shall need to assume the following:

Condition 4: $\alpha = \min(P_0, P_{10}) > 0$.

In order to prove the analogues of Theorem 3.2, 3.4, and 3.5 we define the following process. Consider a new process (the prime process) with identical state and action spaces, with the same cost structure, but with transition probabilities now given by

$$P'_{ij} = \begin{cases} \dfrac{P_{ij}}{1-\alpha} & \text{for } j \neq 0 \\[2ex] \dfrac{P_{i0}-\alpha}{1-\alpha} & \text{for } j = 0 \end{cases} \qquad \text{for all } j = 0,1,..;\ i > 0$$

$$P'_{j} = \begin{cases} \dfrac{P_{j}}{1-\alpha} & \text{for } j \neq 0 \\[2ex] \dfrac{P_{0}-\alpha}{1-\alpha} & \text{for } j = 0 \end{cases} .$$

We shall make use of the following result given by Ross (see [5]):

The optimal $1-\alpha$ discount rule for the prime process is also an optimal policy (in the average-cost sense) for the original process; and $g = \alpha V'_{1-\alpha}(0)$ where $V'_{1-\alpha}(0)$ is the optimal discounted-costs for the prime process and g is the optimal average-cost for the original process. – i.e.,

$$V'_{1-\alpha}(0) = \min_{R'} \psi'(0,1-\alpha,R')$$

$$g = \min_{R} \varphi(i,R) \quad \text{for all } i = 0,1,...$$

The following lemma is needed and its proof is immediate.

Lemma 4.1: If Conditions 1, 2, 3 hold for the original process then they hold for the prime process.

The following analogue of Theorem 3.2 is thus immediate.

Theorem 4.2: Under Conditions 1, 2, 3, 4 there is a control-limit policy which is optimal in the average-cost sense.

Also noting that $1/1-P'_{i0} \sum_{j\neq 0} P'_{ij}\ C(j) = 1/1-P_{i0} \sum_{j\neq 0} P_{ij}\ C(j)$

and $(1 - (1-\alpha))V'_{1-\alpha}(0) = \alpha\ V'_{1-\alpha}(0) = g$ we get the following analogues of Theorems 3.4 and 3.5.

Theorem 4.3: Under Conditions 1, 2, 3, 4 if i is such that

$$1/1-P_{i0} \sum_{j\neq 0} P_{ij}\ C(j) \leq g$$

then there is an average-cost optimal control-limit policy which does not replace at state i. Also if $P_i > 0$ and the inequality is strict then any policy which replaces whenever in state i is not optimal.

Theorem 4.4: Under Conditions 1, 2, 3, 4 if $P_{ij} = 0$ for all $0 < j < i$ $i = 1,2,..$ (pure deterioration) then the control-limit policy which replaces at state i if and only if

$$1/1-P_{i0} \sum_{j\neq 0} P_{ij}\ C(j) > g$$

is optimal in the average-cost sense.

Thus Theorem 4.3 may be interpreted as saying that if the conditional expected cost for the next stage, given that we don't replace and a failure doesn't occur, is no larger than the optimal average cost then

we should not replace. Theorem 4.4 says that, under pure deterioration, this is the only time we should not replace.

5. Computation of Optimal Policy

We have shown that under Conditions 1, 2, 3, 4 the optimal (discount or average-cost) policy exists and has a simple structure. In this section we discuss possible methods for calculating the optimal policy.

We have shown in section 4 that any method of solution for the discounted-cost problem is also a method for the average-cost problem. This is so because the average cost problem can be converted into a discounted-cost problem by defining the prime transition probabilities

$$P'_{ij} = \begin{cases} \dfrac{P_{ij}}{1-\alpha} & j \neq 0 \\[2ex] \dfrac{P_{i0}-\alpha}{1-\alpha} & j = 0 \end{cases} \qquad \text{and} \qquad P'_{j} = \begin{cases} \dfrac{P_{j}}{1-\alpha} & j \neq 0 \\[2ex] \dfrac{P_{0}-\alpha}{1-\alpha} & j = 0 \end{cases}$$

where $\alpha = \min\{P_0, P_{i0}\}$.

The optimal $1-\alpha$ discount policy for the prime process is the optimal average-cost policy for the original process.

Similarly, any method of solution for the average-cost problem is also a method for the discounted-cost problem. We show this by defining a new proces (the star process) with identical state, action, and cost spaces but with transition probabilities now given by

$$P^*_{ij} = \begin{cases} \beta P_{ij} & j \neq 0 \\ 1-\beta+\beta P_{i0} & j = 0 \end{cases} \qquad \text{and} \qquad P^*_{j} = \begin{cases} \beta P_{j} & j \neq 0 \\ 1-\beta+\beta P_{0} & j = 0 \end{cases}$$

Then it is easy to see that Conditions 1, 2, 3, 4 are satisfied in the star process and it again follows from Ross' result (see [5]) that $\varphi^*(0,R) = (1-\beta)\psi(0,\beta,R)$ for any (stationary deterministic) policy R.

Thus the optimal average-cost policy for the star process is a β-optimal discount policy for the original process if this process starts in state 0. However, it is easy to see that this policy will, in most cases, be β-optimal independent of the initial state. This is so, for example, if all states communicate ($P_i > 0$ for all i is sufficient).

Thus any general method for solving the discount-cost problem may be regarded as a general method for solving the average-cost problem and vice versa (assuming, of course, Conditions 1, 2, 3, 4).

We shall now discuss possible approaches for determining the optimal policy. The first two will be discussed in the discounted-cost framework and the last in the average-cost framework.

Method 1: Policy-improvement Method (β-discount)

We initially choose a policy R and calculate $\psi(i,\beta,R)$ for each $i = 0,1..$. We then "improve" R by forming a new policy $\bar{R}$ which which takes action 1 at state i if

$$\sum_{j=0}^{\infty} P_{ij}\ \psi(j,\beta,R) \leq \psi(0,\beta,R) \quad \text{and action 0 if}$$

$$\sum_{j=0}^{\infty} P_{ij}\ \psi(j,\beta,R) > \psi(0,\beta,R)$$

Then it can be shown that $\psi(i,\beta,\bar{R}) \leq \psi(i,\beta,R)$ for all $i = 0,1,..$. We then improve $\bar{R}$, etc. When no further improvement can be made we have the optimal policy.

To take into account the structure of the optimal policy it is suggested that the initial R be chosen to be a control-limit policy. However, though one would hope that the improved policies would also be control-limit policies, this is not necessarily the case.

Method 2: Successive Approximations (β-discount)

Let $B(I)$ = space of all bounded functions on the non-negative integers.

Define the operator $T: B(I) \to B(I)$ by

$$(TU)(i) = C(i) + \beta \min\{U(0), \sum_{j=0}^{\infty} P_{ij} U(j)\} \qquad i > 0$$

$$(TU)(0) = C(0) + \beta \sum_{j=0}^{\infty} P_j U(j)$$

Then $V_\beta = \{V_\beta(i)\}_{i=0}^{\infty}$ is the unique function such that $TV_\beta = V_\beta$; also $\lim T^n U = V_\beta$ for any U (under the supremum norm). Thus V_β may be obtained by successively applying the operator T to any initial vector U. The optimal policy may then be gotten by applying Corollary 3.3.

In order to take advantage of the known structure of V_β (Lemma 3.1) it is suggested that the initial U-vector has the property that

(i) $\{U(i)\}_{i=1}^{\infty}$ is a non-decreasing sequence

(ii) $U(0) = C(0) + \beta \sum_{j=0}^{\infty} P_j U(j)$

It is easy to check that $T^n U$ also has property (i).

Method 3: Analytic Method (average-cost)

We assume that the process starts in state 0 and let T = time at which the process returns to state 0

i.e. $T = \min\{t : t > 0, X_t = 0\}$

Then it is well known that for any (stationary) policy R

$$\varphi(0,R) = \frac{E_R\left[\sum_{t=0}^{T-1} C(X_t) \mid X_0=0\right]}{E_R T}$$

when $E_R T$ is finite by Condition 4.

Let R_i = control-limit policy which replaces whenever in states $i, i+1, \ldots$.

$$\text{Then } g = \min_R \varphi(R) = \min_{i=1,2,..\infty} \frac{E_{R_i}\left[\sum_{t=0}^{T-1} C(X_t) \mid X_0=0\right]}{E_{R_i} T}$$

and it is sometimes analytically possible to determine the above minimum and the minimal value of i. One possible approach, which is sometime applicable, is to treat $\varphi(R_i)$ as a continuous function of i and try to minimize it by using differential calculus.

Example 5.1: Consider a process for which a unit in state i, if not replaced, either remains in state i or fails. Also suppose that the failure probability is independent of the state and also equals the probability that a new unit is defective.

$\therefore$ $P_{ii} = 1 - \alpha$ $\quad P_{i0} = \alpha$ $\quad i = 1,2\ldots$

$P_0 = \alpha$

$$\text{Then } \varphi(R_i) = \frac{C(0) + 1/\alpha \sum_{j=1}^{i-1} P_j C(j) + \sum_{j=i}^{\infty} P_j C(j)}{1 + 1/\alpha \sum_{j=1}^{i-1} P_j + \sum_{j=i}^{\infty} P_j}$$

Now suppose $C(j) = N(1-(1/2)^j) \qquad j = 1,2,\ldots$

$$P_j \doteq (1-\alpha)(1/2)^j \qquad j = 1,2,\ldots$$

$$\therefore \varphi(R_i) = \frac{C(0)+N1-\alpha/\alpha[1-(1/2)^{i-1}-1/3+1/3(1/4)^{i-1}]+N(1-\alpha)[(1/2)^{i-1}-(1/3)(1/4)^{i-1}]}{1 + 1-\alpha/\alpha[1-(1/2)^{i-1}] + (1-\alpha)(1/2)^{i-1}}$$

To find the value of i which minimizes the above we differentiate, set equal to zero, and solve for i.

For $N = 100$, $C(0) = 200$, $\alpha = .1$. The minimum value of i lies between 2 and 3. Thus it seems reasonable that the optimal policy should be either R_2 or R_3.

To check we first convert to the discount (prime) problem.

$$P'_{ii} = 1 \qquad i = 1,2,\ldots$$

$$P'_i = P_i/1-\alpha = (1/2)^i \qquad i = 1,2,\ldots$$

$$\beta = 1-\alpha = .9$$

then $\psi'(0,\beta,R_2) = 1/\alpha\ \varphi(R_2) = 777$

$$\psi'(1,\beta,R_2) = 500$$

$$\psi'(i,\beta,R_2) = 100(1-(1/2)^i) + .9\ \psi'(0,\beta,R_2) \qquad i = 2,3,\ldots$$

$$= 699.6 + 100(1-(1/2)^i)$$

The improved rule is the one which replaces at state i if and only if $\psi'(i,\beta,R_2) > \psi'(0,\beta,R_2)$

$\therefore$ the improved rule is R_3.

Now $\psi'(0,\beta,R_3) = 1/\alpha\ \varphi(R_3) = 770$

$\psi'(1,\beta,R_3) = 500$

$\psi'(2,\beta,R_3) = 750$

$$\left.\begin{aligned}\psi'(i,\beta,R_3) &= 100(1-(1/2)^i) + .9(770)\\ &= 693 + 100(1-(1/2)^i)\end{aligned}\right\} \quad i = 3,4,\ldots$$

$\therefore$ the improved rule is again R_3.

$\therefore$ R_3 is the optimal average-cost policy — i.e., the optimal policy replaces at states $3,4,5,\ldots$ — and the optimal average cost = 77.

Q.E.D.

Example 5.2: By making use of the "star" process and the idea of method 3 we can sometimes get a closed-form expression for the expected discounted costs. Suppose we have (as in example 5.1)

$P_{ii} = 1-\alpha$, $P_{i0} = \alpha \qquad i = 1,2,..$

$P_0 = \alpha$

Let $P^*_{ii} = \beta(1-\alpha)$, $P^*_{i0} = 1-\beta + \alpha\beta$

$P^*_i = \beta P_i \qquad P^*_0 = 1-\beta + \alpha\beta$

then $\psi(0,\beta,R_i) = 1/1-\beta\ \varphi^*(0,R_i)$

$$\text{where } \varphi^*(0,R_i) = \frac{C(0) + 1/\alpha^* \sum_{j=1}^{i-1} P^*_j\, C(j) + \sum_{j=i}^{\infty} P^*_j\, C(j)}{1 + 1/\alpha^* \sum_{j=1}^{i-1} P^*_j + \sum_{j=i}^{\infty} P^*_j}$$

and where $\alpha^* = 1 - \beta + \alpha\beta$.

Q.E.D.

6. Extension to Penalty Costs

One possible extension of the above theory is to include a penalty cost A which is incurred whenever the process goes to state 0 without the replacement action being chosen — i.e. whenever a non-planned replacement occurs. This can be treated by letting the cost function depend not only on the state but also on the action chosen.

Thus let $C(i,j)$ = expected cost incurred when the process is in state i and action j is chosen, $i > 0$, $j = 0,1$.

However, the value of $C(i,j)$ depends upon whether we are in the discount or average-cost case.

Discount Case: In the discount case

$$C(i,1) = C(i) + \beta A P_{i0} \qquad i > 0$$

$$C(i,0) = C(i)$$

This is so since the expected cost of not replacing when in state i includes the operating cost plus an expected penalty cost which is discounted since it is incurred at the next stage. It can be shown, in exactly the same manner as before, that Theorem 3.2 remains true. Similarly Theorems 3.4 and 3.5 can be shown to remain true under the proviso that

$$1/1 \cdot P_{i0}[\sum_{j \neq 0} P_{ij} C(j) + A P_{i0}] \quad \text{replaces}$$

$$1/1\text{-}P_{i0} \sum_{j \neq 0} P_{ij} C(j) \quad \text{in the statement of these theorems.}$$

Average-cost Case: In the average-cost case

$$C(i,1) = C(i) + A P_{i0} \qquad i > 0$$

$$C(i,0) = C(i) \qquad i > 0 .$$

It can then be shown that Theorem 4.2 remains true and that Theorems 4.3 and 4.4 remain true under the proviso that $1/1-P_{i0}\ [\sum_{j\neq 0} P_{ij}\ C(j) + A\ P_{i0}]$ replaces $1/1-P_{i0} \sum_{j\neq 0} P_{ij}\ C(j)$ in the statement of these theorems. This is shown, as before, by reduction to the discount case via the prime problem. Theorems 4.3 and 4.4 follow, for instance, from the discounted results by noticing

$$
\begin{aligned}
\text{that}\quad C(i,1) &= C(i) + A\ P_{i0} \\
&= C(i) + A((1-\alpha)P'_{i0} + \alpha) \\
&= C(i) + A'(1-\alpha)P'_{i0}
\end{aligned}
$$

$$
\text{where}\quad A' = A\left[\frac{(1-\alpha)P'_{i0} + \alpha}{(1-\alpha)P'_{i0}}\right] \quad \text{and} \quad P'_{ij} = \begin{cases} \dfrac{P_{ij}}{1-\alpha} & j \neq 0 \\[2ex] \dfrac{P_{i0} - \alpha}{1-\alpha} & j = 0 \end{cases}
$$

The results then follow by further noticing that

$$
\begin{aligned}
1/1-P'_{i0}[\sum_{j\neq 0} P'_{ij}\ C(j) + A'P'_{i0}] &= 1/1-P_{i0} \sum_{j\neq 0} P_{ij}\ C(j) + A'P'_{i0}/1-P'_{i0} \\
&= 1/1-P_{i0}[\sum_{j\neq 0} P_{ij}\ C(j) + A\ P_{i0}]\ .
\end{aligned}
$$

7. Generalization to Include Stocking of Reserve Units.

It has been assumed up to now that each time a unit fails or is replaced a new unit must be purchased and a cost $C(0)$ is incurred. This cost $C(0)$ includes both the cost of buying, delivering, and installing the new unit and also a cost due to the fact that no machine is in use for a period. We shall now, however, allow for the possibility of purchasing more than one unit and keeping some in reserve. This might be desirable for (1) it might be less expensive to buy the units in quantity, (2) it might cut down on delivery and installation costs, and (3) it would cut down the cost due to idleness (no units in use). The state space will consist of states 0 and (n,i) $i=1,2,\ldots n=0,1,\ldots N-1$. The process will be said to be in state (n,i) when the unit in use is in operating state i and there are n units in reserve. State 0 corresponds to the situation where there are no units in reserve and no (operating) unit in use. If the process is in state (n,i) at time t then one of two possible actions must be chosen. Action 1 leaves the unit in use alone and action 0 replaces it. If the unit in use is in operating state i and action 1 is chosen then the unit will fail before the next observation with probability P_{i0}. If the unit fails, and reserve units are on hand, then the unit will be replaced at the end of the period. When the process is in state 0 there are N possible actions – $a_1,\ldots,a_N$ – where action a_K corresponds to purchasing K new units. Whenever a new (or reserve) unit is put in use then we again assume that there is a probability distribution $\{P_i\}_{i=1}^{\infty}$ over its initial state. However, we shall now suppose that there is zero probability of a new unit being (initially) inoperative. This

condition is not essential but it will simplify notation and it also clarifies the above formulation. The transition probabilities will thus be as follows:

$$P(X_{t+1} = (n,j) | X_t = (n,i),\ \Delta_t = 1) = P_{ij} \qquad 0 \leq n \leq N-1 \quad i,j > 0$$

$$P(X_{t+1} = (n-1,j) | X_t = (n,i),\ \Delta_t = 1) = P_{i0}P_j \quad 1 \leq n \leq N-1 \quad i,j > 0$$

$$P(X_{t+1} = 0 | X_t = (0,i),\ \Delta_t = 1) = P_{i0} \qquad i > 0$$

$$P(X_{t+1} = (n-1,j) | X_t = (n,i),\ \Delta_t = 0) = P_j \qquad 1 \leq n \leq N-1 \quad i,j > 0$$

$$P(X_{t+1} = 0 | X_t = (0,i),\ \Delta_t = 0) = 1 \qquad i > 0$$

$$P(X_{t+1} = (K-1,j) | X_t = 0,\ \Delta_t = a_K) = P_j \qquad 1 \leq K \leq N-1,\ j > 0$$

$$\text{where } \sum_{j=1}^{\infty} P_j = 1, \quad \sum_{j=0}^{\infty} P_{ij} = 1 \qquad i > 0.$$

The costs are as follows: Whenever the process is in state (n,i) an expected cost $C(n,i)$ is incurred. $C(n,i)$ includes the expected operating cost of a unit in state i and the inventory costs involved in holding n units in reserve. When the process is in state 0 and action a_K is chosen then there is a cost $C(K)$ incurred – this includes the cost of buying, installing, and delivering K new units; it may also include a cost due to no unit being in use for a period.

We shall now determine the structure of the optimal policy. In the β-discount case the results follow in an almost identical manner as in section 3. However for the average-cost case the method of section 4 no longer works and a new method of attack is developed.

Consider the following condition:

Condition 1': $\{C(n,i)\}_{i=1}^{\infty}$ is a non-decreasing bounded sequence for each $n = 0,1,\ldots,N-1$.

For any policy R, let $\Psi(Z,\beta,R) = \sum_{t=0}^{\infty} \beta^t E_R[C(X_t,\Delta_t)|X_0 = Z]$ where $Z = 0$, (n,i) $i > 0$, $n \le N-1$; and where

$$C(X_t,\Delta_t) = \begin{cases} C(X_t) & \text{for } X_t \neq 0 \\ C(K) & \text{for } X_t = 0,\ \Delta_t = a_K \end{cases}. \quad \text{Let } V_\beta(Z) = \min_R \Psi(Z,\beta,R).$$

Lemma 7.1: Under Conditions 1',2,3 $\{V_\beta(n,i) - C(n,i)\}_{i=1}^{\infty}$ is a non-decreasing sequence for each $n = 0,1,\ldots,N-1$.

Proof: Same as proof of Lemma 3.1. Q.E.D.

Definition: A policy R is said to be a generalized control-limit policy if there exists integers (possibly infinite) $i_0,\ldots,i_{N-1}$ and an integer K, $1 \le K \le N$, such that R chooses action 0 when in state (n,i) iff $i \ge i_n$ and R chooses action a_K whenever in state 0.

Theorem 7.2: Under Conditions 1',2,3 there is a generalized control-limit policy which is β-optimal.

Proof:

$$V_\beta(n,i) = \min \begin{cases} C(n,i) + \beta \sum_{j\neq 0} P_{ij}\, V_\beta(n,j) + \beta\, P_{i0} \sum_{j=1}^{\infty} P_j V_\beta(n-1,j) \\ C(n,i) + \beta \sum_{j=1}^{\infty} P_j\, V_\beta(n-1,j) \end{cases}$$

The argument now follows as presented in theorem 3.2. Q.E.D.

Thus the β-optimal policy purchases K_0 units when in state 0, and when there are n units in reserve it replaces the unit in use if its operating state is larger than some preassigned number i_n-1.

For any $Z = 0$, (n,i) $i > 0$, $n \leq N-1$, let

$$\varphi(Z,R) = \lim_{m \to \infty} \sum_{t=0}^{m} E_R[C(X_t,\Delta_t) \mid X_0 = Z]$$

A policy R^* is said to be average-cost optimal if

$$\varphi(Z,R^*) = \min_R \varphi(Z,R) \quad \text{for all states } Z.$$

We shall need the following:

<u>Condition 4'</u>: $P_{10} > 0$

<u>Lemma 7.3</u>: Under Conditions 1',2,3,4', for any (stationary) policy R, $M_{Z0}(R) \leq N/P_{10}$ for all states $Z=0$, (n,i) $i > 0$, $n \leq N-1$; where $M_{Z0}(R)$ denotes the mean recurrence time to go from state Z to state 0 when policy R is employed.

Proof: Let Y_t = number of units in reserve at time t. Then since $P_{i0} \geq P_{10}$ for all $i > 0$ it follows that $P_R\{Y_{t+1} = n-1 \mid Y_t = n\} \geq P_{10}$ for any policy R. The lemma follows immediately. Q.E.D.

<u>Theorem 7.4</u>: Under Conditions 1',2,3,4' there exists bounded numbers g, $f(n,i)$ $n = 0,1,\ldots,N-1$, $i > 0$ such that

(i) $\{f(n,i) - C(n,i)\}_{i=1}^{\infty}$ is non-decreasing for each $n=0,1,\ldots,N-1$

(ii) (a) $$g = \min_{1 \leq K \leq n} \{C(K) + \sum_{j=1}^{\infty} P_j f(K-1,j)\}$$

(b) $$g + f(n,i) = C(n,i) + \min \begin{cases} \sum_{j \neq 0} P_{ij} f(n,j) + P_{i0} \sum_{j=1}^{\infty} P_j f(n-1,j) \\ \sum_{j=1}^{\infty} P_j f(n-1,j) \end{cases}$$

(where $f(-1,j) \equiv 0$ for all j)

(ii) (c) The policy R^* which when in state (n,i) takes action 1 when the first term of (b) is minimum and action 0 otherwise, and which when in state 0 selects the action which minimizes (a) is average-cost optimal.

(iii) $\varphi(Z,R^*) = g$ for all states Z.

Proof: The proof of (ii) and (iii) follows from Lemma 7.3 and Theorems 1.1, 1.2, and 1.4 of [5]. (i) follows from Lemma 7.1 and the remark following the proof of Theorem 1.1 of [5]. Q.E.D.

Corollary 7.5: Under Conditions 1',2,3,4' there is a generalized control-limit policy which is average-cost optimal.

Proof: From (ii) of Theorem 7.4 the policy which replaces when in state (n,i) iff

$$1/1-P_{i0} \sum_{j\neq 0} P_{ij}\, f(n,j) > \sum_{j=1}^{\infty} P_j\, f(n-1,j)$$

is average-cost optimal. The result then follows from part (i) of Theorem 7.4 and Lemma 2.1. Q.E.D.

We now prove the analogues of Theorems 4.3 and 4.4

Theorem 7.6: Under Conditions 1',2,3,4' if (n,i) is such that $1/1-P_{i0} \sum_{j\neq 0} P_{ij}\, C(n,j) \leq g$ then there is an average-cost optimal generalized control-limit policy which does not replace at state (n,i).

Proof: From (ii) of Theorem 7.4 we have that

$$g + f(n,j) \leq C(n,j) + \sum_{j=1}^{\infty} P_j\, f(n-1,j)$$

$$\therefore\ g + 1/1-P_{i0} \sum_{j\neq 0} P_{ij}\, f(n,j) \leq 1/1-P_{i0} \sum_{j\neq 0} P_{ij} C(n,j) + \sum_{j=1}^{\infty} P_j f(n-1,j)$$

$\therefore\ 1/1-P_{i0} \sum_{j\neq 0} P_{ij}\ C(n,j) \leq g \Rightarrow 1/1-P_{i0}\sum_{j\neq 0} P_{ij} f(n-1,j) \leq \sum_{j=1} P_j f(n-1,j)$

The Theorem follows from Corollary 7.5 Q.E.D.

Theorem 7.7: Under Conditions 1',2,3,4' if $P_{ij} = 0$ for all $0 < j < i$ (pure deterioration) then the generalized control-limit policy which replaces at state (n,i) iff

$1/1-P_{i0} \sum_{j\neq 0} P_{ij}\ C(n,j) > g$ is average-cost optimal.

Proof: Suppose that $1/1-P_{i0} \sum_{j\neq 0} P_{ij}\ C(n,j) > g$ and suppose further that

$$1/1-P_{i0} \sum_{j\neq 0} P_{ij}\ f(n,j) \leq \sum_{j=1}^{\infty} P_j\ f(n-1,j)$$

$$\begin{aligned}
\therefore\ g + f(n,i) &= C(n,i) + \sum_{j\neq 0} P_{ij}\ f(n,j) + P_{i0} \sum_{j=1}^{\infty} P_j\ f(n-1,j) \\
&= C(n,i) + \sum_{j\neq 0} P_{ij}\ C(n,j) + \sum_{j\neq 0} P_{ij}(f(n,j) - C(n,j)) \\
&\quad + P_{i0} \sum_{j=1}^{\infty} P_j\ f(n-1,j) \\
&> C(n,i) + g(1-P_{i0}) + (1-P_{i0})(f(n,i) - C(n,i)) \\
&\quad + P_{i0} \sum_{j=1}^{\infty} P_j\ f(n-1,j)
\end{aligned}$$

$$\therefore\ g + f(n,i) > C(n,i) + \sum_{j=1}^{\infty} P_j\ f(n-1,j)$$

which is a contradiction by Theorem 7.4 (ii)

$$\therefore\ 1/1-P_{i0} \sum_{j\neq 0} P_{ij}\ f(n,j) > \sum_{j=1}^{\infty} P_j\ f(n-1,j)$$

$\therefore$ The result follows from Corollary 7.5 and Theorem 7.6. Q.E.D.

Corollary 7.8: Under the conditions of Theorem 7.7 if $C(n,i)$ is monotone non-decreasing in n for i fixed then there is an optimal generalized control-limit policy such that $i_0 \geq i_1 \geq \cdots i_{N-1}$.

Proof: Follows from Theorem 7.7.

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The following model is considered: A machine is inspected at the beginning of discrete time periods and its operating state (condition) is determined. After each inspection a decision upon whether or not to replace the machine at the end of the period must be made. It is shown that under certain conditions on the costs and the transition probabilities the optimal policy has a simple form. A generalization to the case where one is allowed to stock reserve machines is also given.

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